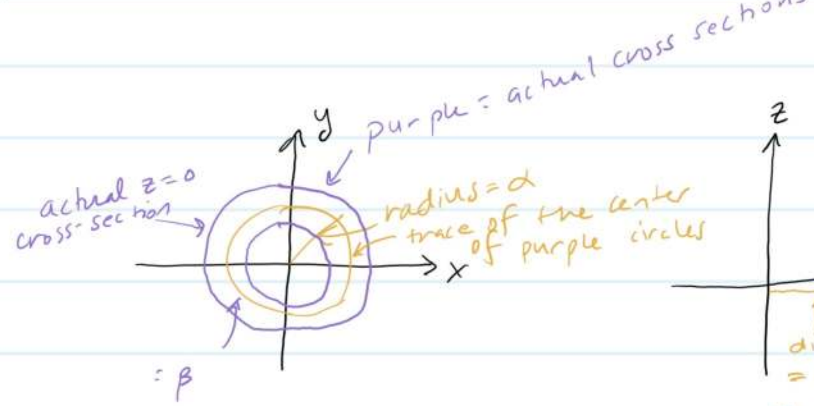
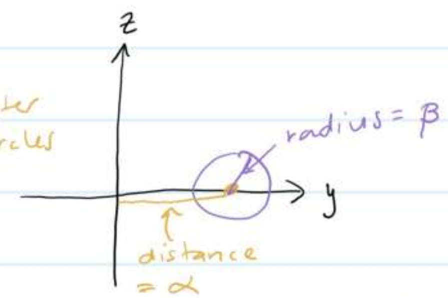
Surfaces & Calculus

11/22/21

Recall: A surface is $\vec{s}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ on domain D.

Ex: The torus w/ major radius α & minor radius β (for $\alpha > \beta > 0$) is the surface with equation
$\vec{s}(u,v) = \langle (\alpha + \beta\cos(u))\cos(v), (\alpha + \beta\cos(u))\sin(v), \sin(u) \rangle$ on domain $D = [0, 2\pi] \times [0, 2\pi]$





$r = \alpha + \beta\cos(\theta)$ is equation of purple circle in polar coordinates

I. Tangent Planes

The tangent plane to surface $\vec{s}(u,v)$ at input point (a,b) has normal vector $\vec{n}(a,b) = \vec{s}_u(a,b) \times \vec{s}_v(a,b)$ where $\vec{S}_u = \frac{\partial \vec{s}}{\partial u} = \langle x_u, y_u, z_u \rangle$ ← partial derivative; ↑ vector of partial derivatives

Ex: Compute the tangent plane to the torus with major radius (4) & minor radius (1) at point $\vec{s}\left(\frac{3\pi}{4}, \frac{\pi}{4}\right)$

Solution: We want $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$, & we're given $\vec{p} = \vec{s}\left(\frac{3\pi}{4}, \frac{\pi}{4}\right)$

$\vec{s}(u,v) = \langle (4 - \cos(u))\cos(v), (4 + \cos(u))\sin(v), \sin(u) \rangle$

$\therefore \vec{s}\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) = \left\langle \left(4 + \cos\left(\frac{3\pi}{4}\right)\right)\cos\left(\frac{\pi}{4}\right), \left(4 + \cos\left(\frac{3\pi}{4}\right)\right)\sin\left(\frac{\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right) \right\rangle$ on $[0, 2\pi]^2$

$\therefore \vec{p} = \vec{s}\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) = \left\langle \left(4 + \cos\left(\frac{3\pi}{4}\right)\right)\cos\left(\frac{\pi}{4}\right), \left(4 + \cos\left(\frac{3\pi}{4}\right)\right)\sin\left(\frac{\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right) \right\rangle$

$= \langle (4-\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}), (4-\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}), \frac{1}{\sqrt{2}} \rangle = \langle \frac{4}{\sqrt{2}}-\frac{1}{2}, \frac{4}{\sqrt{2}}-\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$

(comment: we just need normal at $(\frac{3\pi}{4}, \frac{\pi}{4})$, but we'll compute it more generally for use later on...)

$\vec{n} = \vec{s}_u \times \vec{s}_v$

$\vec{s}_u = \langle -\sin(u)\cos(v), -\sin(u)\sin(v), \cos(u) \rangle$

$\vec{s}_v = \langle -\sin(v)(4+\cos(u)), \cos(v)(4+\cos(u)), 0 \rangle$

$\vec{n}(u,v) = \vec{s}_u(u,v) \times \vec{s}_v(u,v)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(u)\cos(v) & -\sin(u)\sin(v) & \cos(u) \\ -\sin(v)(4+\cos(u)) & \cos(v)(4+\cos(u)) & 0 \end{vmatrix}$$

$= (0-\cos(u)\cos(v)(4+\cos(u)))\vec{i} - (0-\cos(u)(-\sin(v)(4+\cos(u))))\vec{j}$
$+ (-\sin(u)\cos(v)\cos(v)(4+\cos(u)) - \sin(u)\sin(v)\sin(v)(4+\cos(u)))\vec{k}$

$= -\cos(u)\cos(v)(4+\cos(u))\vec{i} - \cos(u)\sin(v)(4+\cos(u))\vec{j}$
$- \sin(u)(4+\cos(u))(\cos^2(u)+\sin^2(v))\vec{k}$

$= -(4+\cos(u)) \langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(u) \rangle$

normal vector to the torus at <u>every</u> input point (u,v)

$\therefore \vec{n}(\frac{3\pi}{4}, \frac{\pi}{4}) = -(4+\cos(\frac{3\pi}{4})) \langle \cos(\frac{3\pi}{4})\cos(\frac{\pi}{4}), \cos(\frac{3\pi}{4})\sin(\frac{\pi}{4}), \sin(\frac{3\pi}{4}) \rangle$

$= -(4-\frac{1}{\sqrt{2}}) \langle -\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= -(4-\frac{1}{\sqrt{2}}) \langle -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle$

$\therefore$ The tangent plane to this torus at $\vec{s}(\frac{3\pi}{4}, \frac{\pi}{4})$ is $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$

i.e. $\vec{n}(\frac{3\pi}{4}, \frac{\pi}{4}) \cdot (\vec{x} - \vec{s}(\frac{3\pi}{4}, \frac{\pi}{4})) = 0$

i.e. $-(4-\frac{1}{\sqrt{2}}) \langle -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \rangle \cdot \langle x - \frac{4}{\sqrt{2}} + \frac{1}{2}, y - \frac{4}{\sqrt{2}} + \frac{1}{2}, z - \frac{1}{\sqrt{2}} \rangle = 0$

i.e. $-\frac{1}{2}\left(x-\frac{4}{\sqrt{2}}+\frac{1}{2}\right)-\frac{1}{2}\left(y-\frac{4}{\sqrt{2}}+\frac{1}{2}\right)+\frac{1}{\sqrt{2}}\left(z-\frac{1}{\sqrt{2}}\right)=0$ ▨

II. Surface Area:

The surface area of the surface S parameterized by $\vec{s}(u,v)$ on domain D is $\text{Area}(S)=\iint_D |\vec{s}_u \times \vec{s}_v|\, dA$

↑ area of parallelogram det'd by $\vec{s}_u$ & $\vec{s}_v$ at any given point

Q: why that formula?

A: Piecewise linearly approximate surface S via parallelograms. Limit sums of those area approximations

(see calc class website for a GeoGebra sheet w/ approximations...)

Ex: For the torus w/ major radius (4) & minor radius (1), compute the surface area.

solution: $\text{Area}(S)=\iint_D |\vec{s}_u \times \vec{s}_v|\, dA$

from before $\vec{s}_u(u,v)\times\vec{s}_v(u,v) = -(4+\cos(u))\langle \cos(u)\cos(v), \cos(u)\sin(v), \sin(u)\rangle$

so we compute:

$$|\vec{s}_u \times \vec{s}_v| = |-(4+\cos(u))|\sqrt{\cos^2(u)\cos^2(v)+\cos^2(u)\sin^2(v)+\sin^2(u)}$$

$$= |4+\cos(u)|\sqrt{\cos^2(u)\underbrace{(\cos^2(v)+\sin^2(v))}_{1}+\sin^2(u)}$$

$$= 4+\cos(u)$$

$D=[0,2\pi]^2$ only traverses the torus once...

$$\therefore \text{Area}(S) = \iint_D |\vec{s}_u \times \vec{s}_v|\, dA = \int_{u=0}^{2\pi}\int_{v=0}^{2\pi}(4+\cos(u))\,dv\,du$$

$$= \int_{u=0}^{2\pi}(4+\cos(u))\Big[v\Big]_{v=0}^{2\pi}du = 2\pi\int_{u=0}^{2\pi}(4+\cos(u))\,du$$

$$= 2\pi\Big[4u+\sin(u)\Big]_{u=0}^{2\pi} = 2\pi[4(2\pi-0)+(0-0)] = 2\pi(8\pi) = 16\pi^2$$

▨

Exercise: Compute the surface of the general torus w/ major radius α & minor radius β $(\alpha > \beta > 0)$ (Result should be $4\alpha\beta\pi^2$)

III. Surface Integrals

The (surface) integral of function $f(x,y,z)$ over surface S parameterized by $\vec{s}(u,v)$ on Domain D is:

$$\iint_S f \, dS = \iint_D f(\vec{s}(u,v)) \, |\vec{s}_u \times \vec{s}_v| \, dA$$

Q: Why this formula?

A1 (analogy w/ line integrals) $\int_C f \, d\vec{r} = \int_{dom(\vec{r})} f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$

A2 (analogy w/ area integrals): In the double integrals case:

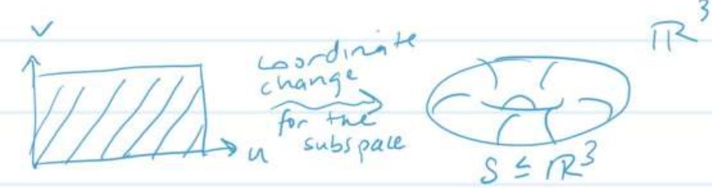
$$\text{Area}(R) = \iint_R 1 \, dA$$

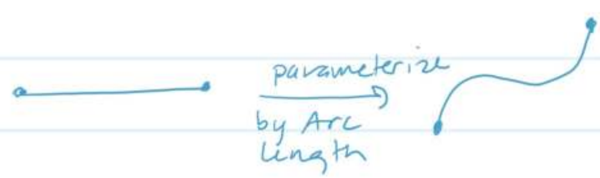
In surfaces, $\text{Area}(S) = \iint_S 1 \, dS$

$$= \iint_{dom(\vec{s})} |\vec{s}_u \times \vec{s}_v| \, dA$$

↖ expect this behavior

NB: The correct, rigorous way to understand "$dS = |\vec{s}_u \times \vec{s}_v| \, dA$" is via a Jacobian...





Actually: $|\vec{s}_u \times \vec{s}_v|$ is the Jacobian of a coordinate change